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LETTER TO THE EDITOR

Generation and monitoring of a discrete stable random process

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Abstract

A discrete stochastic process with stationary power law distribution is obtained from a death-multiple immigration population model. Emigrations from the population form a random series of events which are monitored by a counting process with finite-dynamic range and response time. It is shown that the power law behaviour of the population is manifested in the intermittent behaviour of the series of events.

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The connectivity of complex networks [1] and microdynamical transport properties of sandpile cellular automata [2] are aspects of complex systems that are described by discrete random variables with probability distribution functions (PDFs) having power law tails. The continuous analogue of such random variables are the stable or Lévy distributions [3], and these can model fluctuations in diverse non-equilibrium physical [4], biological [5] and financial [6] complex systems. Such distributions describe fluctuations of a scale-free or fractal nature, the moments for which do not generally exist. Whilst instances of single fold random variables with these characteristics abound, stationary random *processes* describing these fluctuations have not, hitherto, been derived using techniques of classical stochastic processes. Here, a discrete Markov stochastic process with a 'stable' law is derived by generalizing a population model originally motivated by non-classical photon fluctuation phenomena [7, 8]. Emigrations from the population form a series of events that are monitored with a counting process of finite-dynamic range and response time. The counting process is modelled by 'clipping' the time series, saturating naturally events occurring in the tail of the distribution and thereby regularizing the fluctuations so that the moments of their probability distributions exist. Statistical measures of the clipped time series, such as its mean, autocorrelation and the PDF for times to the first count and between counts, nevertheless display characteristics of the scale-free nature of the parent population.

The population evolves according to a death-multiple immigration model. The population size increases through immigration of singles, pairs, ... m-tuplets ..., which arrive at rates

$\alpha_m \geqslant 0$, and is depleted at a constant rate μ by deaths that occur in proportion to the instantaneous size of the population. The rate equation for this process,

$$\frac{\mathrm{d}P_N(t)}{\mathrm{d}t} = \mu(N+1)P_{N+1} - \mu N P_N - P_N \sum_{m=1}^{\infty} \alpha_m + \sum_{m=1}^{N} \alpha_m P_{N-m} \quad (1)$$

describes the evolution of $P_N(t)$, the probability that the population comprises N members at time t. This model is at once a simplification and generalization of known processes. The situation when births are incorporated into the above and the immigrants arrive singly leads to the birth–death–immigration (BDI) process [9]. The stationary solution for this process is the negative binomial distribution, and for a specific choice of parameters the model describes the Bose–Einstein fluctuations of thermal light. Properties of this process have intermediate power law regimes for a range of parameters [10], but the single fold stationary distribution does not have the characteristic scale-free property that is a hallmark of the complex behaviour. The BDI model also describes the laser below threshold, where stimulated and spontaneous emission of photons in a cavity is analogous to birth and immigration, and absorption corresponds to deaths. Another variation of the process given by equation (1) was motivated by modelling non-classical or 'squeezed' light [8] and obtains when the immigrants arrive in pairs [7]. Whilst the correlation properties of this and the BDI processes are similar, the stationary state differs markedly, exhibiting odd–even effects.

The process described by equation (1) has no births, but extinction of the population is prevented by the continual arrival of immigrants. The solution of equation (1) can be found using the generating function $Q(s;t) = \langle (1-s)^N \rangle = \sum_{N=0}^{\infty} (1-s)^N P_N(t)$ from which factorial moments and probabilities can be determined [7]:

$$\frac{\langle N(N-1)(N-2)\cdots(N-r+1)\rangle}{\langle N\rangle^r} = \left(-\frac{\mathrm{d}}{\mathrm{d}s}\right)^r Q(s;t)\bigg|_{s=0}$$

$$P_N(t) = \frac{1}{N!}\left(-\frac{\mathrm{d}}{\mathrm{d}s}\right)^N Q(s;t)|_{s=1}. \quad (2)$$

The single fold generating function satisfies the partial differential equation

$$\frac{\partial Q}{\partial t} = -\mu s \frac{\partial Q}{\partial s} + \left[\sum_{m=1}^{\infty} \alpha_m((1-s)^m - 1)\right] Q$$

with boundary conditions $Q(0,t) = 1$, and $Q(s;0) = Q_0(s) = (1-s)^M$ which imply the probability distribution has unit normalization at all times and that the population initially has M members present. Consider the specific choice of coefficients $\alpha_m = -a\Gamma(m-\nu)/(\Gamma(-\nu)m!)$. These are all positive only if $0 < \nu < 1$ and yield the PDE

$$\frac{\partial Q}{\partial t} = -\mu s \frac{\partial Q}{\partial s} - a s^\nu Q$$

with solution

$$Q(s;t) = Q_0(s\exp(-\mu t))Q_E(s;t) = (1 - s\exp(-\mu t))^M \exp\left[-\frac{as^\nu}{\nu\mu}(1 - \exp(-\nu\mu t))\right]. \quad (3)$$

Ostensibly the stationary solution $Q(s;\infty) = Q_{st}(s) = \exp(-as^\nu/\nu\mu)$ has similar structure to the characteristic function of the continuous stable distributions, although $Q_{st}(s)$ is the generating function of a discrete distribution, giving a power law tail $P_N \sim N^{-(1+\nu)}$ with index in the range -1 to -2. Note that the case $\nu = 1$ does not have a power law tail but rather generates the Poisson distribution. Figure 1(a) shows an example of the evolution to the

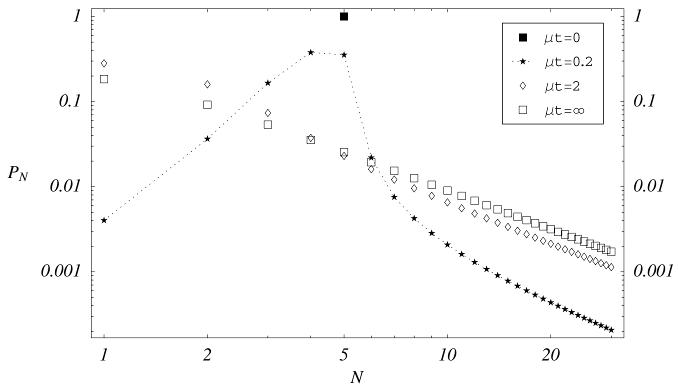

Figure 1. Temporal evolution of the PDF of the death-multiple immigration population model initiated with $M = 5$ with $\nu = 1/2$, $a = 1$ and death rate $\mu = 2$ for times $\mu t = 0$ (■) , 0.2 (⋆), 2 (♦) and ∞ (□). The $N^{-3/2}$ tail of the distribution is established immediately.

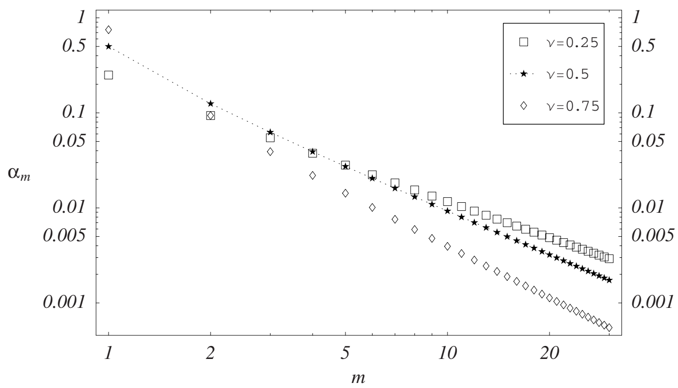

Figure 2. Immigration rates α_m as function of the order of the m-tuplet for $\nu = 1/4$, 1/2 and 3/4.

stationary PDF from the initial state $P_N(0) = \delta_{N,M}$ with $M = 5$ for when $\nu = 1/2$ and $\mu = 2$.[1] The tail of the distribution is established immediately, implying the moments do not exist for any $t > 0$. This is because the rates α_m permit a large number of immigrants to enter the population with probability that falls off like a power law. Figure 2 shows the coefficients α_m for values $\nu = 1/4, 1/2$ and 3/4, and these have an inverse power law dependence on m, $\alpha_m \sim 1/m^{1+\nu}$. Models describing the scale-free growth of the WWW are predicated upon preferential attachment, whereby highly connected sites attract more connections than those less well connected [1]. The rate at which connections are added to the system could then have a power law asymptote that is similar to the α_m. It should be stressed, however, that the form adopted by the rates is not necessarily reflected by the stationary solution of the process. For example, the geometrical or thermal model $\alpha_m = \xi^m$ with $0 < \xi < 1$ is *one* member of the negative binomial class of distributions, but immigrations described by these rates generate the *entire* negative binomial class, $Q_{st}(s) = (1 + \bar{N}s/\beta)^{-\beta}$ with $\bar{N} = \xi/\mu(1-\xi)^2$ and $\beta = 1/\mu(1-\xi)$ [11].

[1] The PDF for the case $\nu = 1/2$ has a closed form solution with stationary law $P_N = 2\pi^{-1/2}(a/\mu)^{N+1/2} \times K_{N-1/2}(2a/\mu)$, where $K_\nu(z)$ is a modified Bessel function [13].

The Markovian nature of the death-multiple immigration process implies that the joint generating function, which describes the population having sizes N and N' following a separation time t, can be deduced from the stationary solution together with equation (3), conditioned upon there being N members initially present, namely

$$
\begin{aligned}
Q(s, s'; t) &= \langle (1-s)^N (1-s')^{N'} \rangle = \sum_{N'=0}^{\infty} Q^{(N)}(s; t) P_{N'} (1-s')^{N'} \\
&= \exp\left[-\frac{a}{\nu\mu} (s^\nu (1 - \exp(-\nu\mu t)) + (s' + (1-s')s\exp(-\mu t))^\nu) \right] \qquad (4)
\end{aligned}
$$

from which joint distributions and, in principle, autocorrelation and higher order statistical measures can be obtained. However, because the joint probabilities also have power law tails, the autocorrelation function is not defined. A noteworthy property of equation (4) is that it is not invariant to the interchange of s with s', which implies that the death-multiple immigration population model does not possess a doubly stochastic representation. Hence, the population cannot be regarded as evolving in response to a continuous random fluctuation: in quantum optics terms [8], the process is non-classical. Equations (3) and (4) provide a closed form solution for the single fold and joint evolution of a stochastic process with discrete power law stationary distribution. The question now arises as to how the fluctuations and their evolution can be measured given that moments of the population do not exist.

There are circumstances where it may be impossible to make a direct measurement of the population but rather only some external manifestation of its evolution. Many experimental situations externally monitor evolution by counting the number n of emigrants that leave the population at rate η. These emigrations form a series of events that can be counted in time intervals of duration T, the integration time. Formulating the externally monitored counting process necessitates introducing the joint probability distribution $p_N(n; T)$ for N individuals present in the stationary population, and with n emigrants having been counted in the integration time interval $[0, T]$. Evolution of the monitored time series follows by modelling the counted emigrants through an additional death process [12], requiring the inclusion of terms $\eta(N+1)p_{N+1}(n; T) - \eta N p_N(n; T)$ to the RHS of equation (1). This new rate equation is solved with the aid of the joint counting generating function $Q_c(s, z; T) = \langle (1-s)^N (1-z)^n \rangle$ that satisfies the PDE

$$
\frac{\partial Q_c}{\partial T} = (\eta z - \bar{\mu} s) \frac{\partial Q_c}{\partial s} - a s^\nu Q_c.
$$

The solution of this equation is initiated from the stationary state of the population, so that $Q_c(s, z; 0) = Q_{st}(s)$ and

$$
\begin{aligned}
Q_c(s, z; T) = Q_{st}(\Phi) \exp\biggl(&\frac{-a}{(1+\nu)\eta z} (\Phi^{\nu+1} F(1+\nu, 1, 2+\nu; \bar{\mu}\Phi/\eta z) \\
&- s^{\nu+1} F(1+\nu, 1, 2+\nu; \bar{\mu} s/\eta z)) \biggr)
\end{aligned}
$$

with $\Phi(s, z; T) = [\eta z + (\bar{\mu} s - \eta z) \exp(-\bar{\mu} T)]/\bar{\mu}$, $\bar{\mu} = \mu + \eta$ a composite death rate, and $F(a, b, c; x)$ the hypergeometric function [13]. Denoting $Q_c(z; T) = Q_c(0, z; T)$ in conjunction with equation (2) and differentiating with respect to the z variable obtains factorial moments and distribution for the counted population. The counting distribution also possesses a power law tail for all integration times T, an observation that distinguishes this integrated process from those with finite mean, which necessarily become Poissonian in the large T limit. Using the method of [14] enables time series for the process to be obtained, one realization

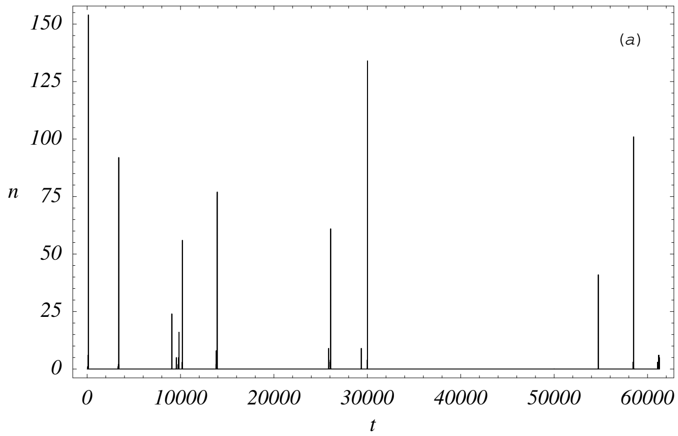

Figure 3. A realization of the counted series of events with $\bar{\mu} = 2$, with other parameters identical to figure 1. The integration time $T = 5$. The emanations show wide variability in the size of the events and also in the times of their occurrence.

being illustrated in figure 3 with the same parameters as for figure 1. Note the variation in size and the intermittent nature of the emissions, both being manifestations of the power law PDF.

Any counting method saturates in excess of its dynamic range. This limitation can be idealized by clipping the time series, an extreme form of which is hard limiting [15] which forms the binary stream:

$$c(t, T) = \begin{cases} 0 & n(t, T) = 0 \\ 1 & n(t, T) > 0 \end{cases}$$

the statistics for which are well defined irrespective of divergent moments in the underlying monitored population. The mean of the clipped counts $\bar{c}$ is finite and higher moments $\langle c^r \rangle = \bar{c}$. Whilst clipping suppresses size variations, the characteristic intermittency is retained, and this property contains information about the power law PDF of the population. The generating function of the clipped counting distribution $Q_{cl}(z; T)$ is particularly simple since the data comprise only 0's and 1's:

$$Q_{cl}(z; T) = p(0, T) + (1 - z)(1 - p(0, T)) = Q_c(1; T) + (1 - z)(1 - Q_c(1; T)).$$

The mean $\bar{c}(T) = 1 - Q_c(1; T)$ depends on the integration time, and for small T this does not increase linearly but rather has the power law dependence, $\bar{c}(T) \sim (a/\nu\bar{\mu})(\eta T)^{\nu} + O((\eta T)^{2\nu})$, indicating that even the most severely limited integrated counting measurements retain a vestige of the scale-free behaviour of the monitored population. For large integration times the clipped mean saturates at unity illustrated in figure 4(a). The clipped autocorrelation function can be found using the joint distribution for counting one emanation in two non-overlapping integration periods of length T, separated by an interval Δt [7, 14]. For $\eta T \ll 1$ and zero delay time,

$$\lim_{\Delta t \to 0} \frac{\langle c(0, T) c(T + \Delta t, T) \rangle}{\bar{c}(T)^2} = \frac{2\bar{\mu}\nu}{a} \left(1 - \frac{1}{2^{1-\nu}} \right) (\eta T)^{-\nu}$$

whose divergence for small T indicates the non-existence of the moments of the parent population.

Another measure of the time series is $w_0(\tau)$, the probability density for the time τ to the first count [7]:

$$w_0(\tau) = -\frac{\partial Q_{cl}(1; \tau)}{\partial \tau} = -\frac{\partial Q_c(1; \tau)}{\partial \tau}$$

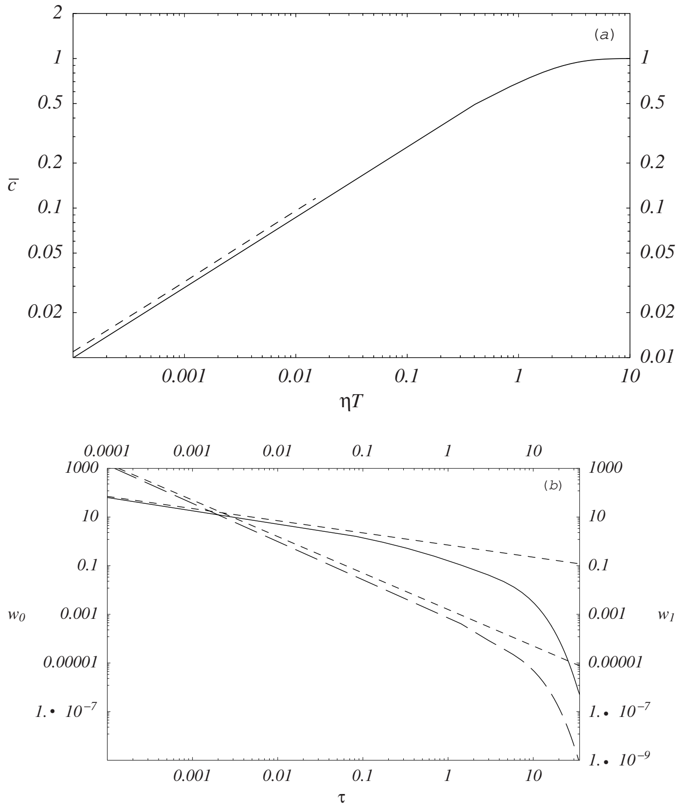

Figure 4. (*a*) Illustrating the dependence of the clipped mean on the integration time T. The dashed line shows the power law asymptote. Parameters used are the same as for figure 3. (*b*) Probability density for the time to the first and inter counting events shown by the full and long-dashed lines respectively, together with their asymptotes shown by short-dashed lines. The parameters are as for figure 3 and the inner scale $\tau_0 = 0.000\,01$.

shown in figure 4(*b*). At small times this has asymptote $w_0(\tau) \sim a(\eta/\bar{\mu})^\nu(\bar{\mu}\tau)^{-(1-\nu)}$, but for large times the tail of the distribution is exponential reflecting the finite-correlation time of the fluctuations. The scaling of $\bar{c}$ with T implies that the apparent rate of occurrence of events $\bar{c}/T$ increases with increasing resolution. This observation necessitates introducing an inner time scale τ_i below which no counts are recorded, whereupon the probability density $w_1(\tau)$ for the times τ between events [16] can be defined:

$$w_1(\tau) = \langle\dot{\tau}(\tau_i)\rangle^{-1}\frac{\partial^2 Q_c(1;\tau)}{\partial\tau^2} = \frac{\tau_i}{\bar{c}(\tau_i)}\frac{\partial^2 Q_c(1;\tau)}{\partial\tau^2} = \frac{1}{w_0(\tau_i)}\frac{\partial^2 Q_c(1;\tau)}{\partial\tau^2}, \qquad \tau_i \leqslant \tau < \infty$$

and is illustrated in figure 4(*b*).

This letter has introduced a discrete stochastic population process governed by births and multiple immigrations. The stationary probability distribution is the discrete analogue of the continuous stable probability densities for a particular choice of immigration rates. Emigrations from this population form a series of discrete events that can be monitored by a counting process of finite-dynamic range and response time. This enables finite measures of integrated statistics to be defined that have embedded power law regimes at small integration times. The power law PDF of the discrete population is transferred to the intermittent behaviour of the series of events. The population model described here can easily be generalized to include births, obtaining a different process that can nevertheless have similar stationary states, however the temporal evolution would be different and could be distinguished by the measurement process. The detection methodology suggests generic means for characterizing intermittent time series that exploit properties of the underlying stochastic process rather than employing taxonomic classification by PDFs alone. An important application of this type of Markovian multiple immigration model will be to provide a simple means of generating temporal data possessing a wide range of stochastic behaviour [11], including those describing complex systems.

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